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Abstract

There has been considerable work recently on the problem of reconstruction of 3D point sets from two images, taken by uncalibrated cameras. However, the point correspondence has to be given. Here we deal with reconstruction of curves rather than points. While we need the correspondence between curves, this is an easier problem because curves are far fewer and more distinctive than points. We derive a simple and general reconstruction method, based on an invariant coordinate system. We then apply it to non-coplanar conics and to combinations of a 3D conic with points. 3D cubics are also discussed. Unlike previous work, we do not need to know the epipolar geometry; we recover it from the images.

Keywords: object recognition, invariants, deformation.

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1. Introduction

The problem of 3D reconstruction from uncalibrated cameras is closely related to the problem of finding projective and affine invariants. This is because, regardless of the camera parameters, we can assume that the camera performs a simple geometric projection (assuming an aberration-free lens). The invariants of this projection are thus independent of the camera parameters. The development of applications of invariants in vision generally (e.g. [13–15]) has thus led to the development of such applications for uncalibrated cameras, e.g. [1–12].

There are no invariants of a projection from 3D to a single 2D image, so we need at least two views. Given the projections of the same 3D object in the two images, the object can be reconstructed. In most of the above mentioned work, the objects are points. One needs at least seven corresponding points for the reconstruction, or eight if the problem is to be solvable linearly. Finding point correspondences is far from a solved problem in vision, and this limits the usefulness of these methods.

The objects we deal with in this paper are curves rather than points. Finding correspondence between curves is much easier than between points. There are far fewer curves in an image than there are points, so a combinatorial explosion is unlikely to be a problem. In addition, unlike points, curves have features that distinguish one from another and can be used to narrow down the possible matches. A few of the references above deal with conic curves but require *a priori* knowledge of the epipolar geometry, or the matching properties of the images. Here we recover this information from the images.

It is easy to calculate the minimal number of parameters the object's description has to have for the problem to be solvable. In the general projective case, each camera projection from 3D to 2D is a 3×4 matrix, having 11 essential coefficients. If the object has n parameters in a known coordinate system, we have $n+22$ unknowns. Of these, we give up 15 by using an indeterminate 3D coordinate system which we cannot recover. This is a result of not knowing the camera parameters. Thus we need to find $n+7$ unknowns. To solve this problem we thus need at least $n+7$ known quantities. If the projection of the object has m measurable parameters in each image, we must thus have

$$2m \geq n + 7$$

In the affine case, the projection matrix has 8 parameters and the 3D coordinate system has 12, so similar arguments yield

$$2m \geq n + 4$$

We see immediately that seven 3D points satisfy the above formula in the projective case, with a total number of $n = 21$ parameters in 3D and $m = 14$ in the 2D projections. In the affine case four points suffice. (However, due to noise, many more corresponding points in the images are actually needed.) A set of two conics in 3D has 16 parameters, projected into ten on each image. Thus such a set will satisfy our requirement in the affine case, but will leave three parameters unknown in the projective case. A cubic curve has 12 parameters in 3D, projected into eight parameters in 2D; thus it too will satisfy the affine requirement but will leave three parameters unknown in the projective case.

For recognizing the objects, only some of the quantities solved for will be useful. These will be object descriptors which are invariant to a 3D transformation. (The rest of the quantities will be related to the cameras). An object with n parameters will have $n - 15$ invariants in the projective case and $n - 12$ in the affine case. The pair of 3D conics thus has one invariant in the projective case and four in the affine case. The cubic has no 3D invariants and therefore cannot be recognized. In other words, all 3D cubics are affine (and projective) equivalents; but we can still derive some information from them about the cameras and the correspondence between the two images. Higher order curves can in theory also be reconstructed, but they are difficult to extract from the images. A more practical approach for a general curve can be to approximate it as a series of conic pieces.

Combinations of curves and points are also of interest. For instance, one conic and two points in 3D have two invariants in the affine case.

In this paper we solve the reconstruction problem in two stages:

1) We recover the epipolar geometry, namely the matching properties between the two images.

2) We find the 3D invariants of the objects, based on the matched images. This is done in an invariant coordinate system which makes the process very simple and general.

2. Finding the Epipolar Geometry

In this section we find the epipolar geometry, namely the matching properties between the images. This will enable us to reconstruct the objects in the next section.

We have two cameras with optical centers at $\mathbf{O}, \mathbf{O}'$ in 3D. The line connecting these points intersects the first image at $\mathbf{o}$ and the second image at $\mathbf{o}'$. These are the epipoles. In other words, an epipole is the projection, in one camera's image, of the other camera's optical center. An arbitrary point $\mathbf{X}$ in 3D defines an epipolar plane $\mathbf{OO'X}$ going through it and the optical centers. This plane intersects the first image in a line $\mathbf{ox}$ and the second image in a line $\mathbf{o'x'}$. These are the epipolar lines. They form in each image a pencil of lines meeting at the epipole. Obviously, for a given $\mathbf{X}$, the epipolar lines match each other since they belong to the same plane. Finding the epipolar geometry means finding the points $\mathbf{o}, \mathbf{o}'$ and finding the matching between the epipolar lines.

The epipolar geometry can be succinctly represented by the so called fundamental matrix. In the case of points, it is quite easy to find this matrix, given the point correspondence. In our case of curves, we did not find this matrix particularly useful; we therefore use a different method.

Our method is based on invariants of the epipolar lines that are equal in the two images. These can be found as follows. The points $\mathbf{X}_i$ form planes $\mathbf{OO'X}_i$ having a common line $\mathbf{OO'}$. Thus we can define a cross ratio of these planes, in the projective case, or a ratio in the affine case. These planes intersect the first image in the epipolar lines $\mathbf{ox}_i$, and therefore these lines have the same (cross) ratio as the planes on which they lie. Similarly, in the other image, the matching epipolar lines $\mathbf{o'x'}_i$ have the same (cross) ratio as their planes. Therefore, *the (cross) ratio of matching epipolar lines is the same in both images.*

This invariance can be used to find the epipoles as follows. Given some matching points $\mathbf{x}_i, \mathbf{x}'_i$, we can express the epipolar lines going through them as functions of the unknown epipoles $\mathbf{o}, \mathbf{o}'$, since these lines go through the epipoles. By equating the ratios of matching epipolar lines, we obtain enough equations to determine $\mathbf{o}, \mathbf{o}'$. For matching curves, it is easy to find matching epipolar lines by drawing tangents to the curves from the epipoles. Since these tangents are projections of the corresponding tangent planes to the 3D curves, they will match. The contact points of the tangents with the curves will also match. We can then use the method described above, i.e. use the ratios of these tangents, to write equations for finding the epipoles.

In the affine case, the epipoles are at infinity, meaning that all epipolar lines in each image are parallel. Thus all epipolar lines in the first image have the same slope s, while those in the second image have a slope s'. The problem of finding the epipoles is now reduced to finding the slopes s, s'.

We will now perform the calculation in the affine case for various configurations. We can represent a family of parallel lines having a common slope s as a vector in homogeneous coordinates

$$\mathbf{l} = (s, 1, d)$$

where d is a distance parameter proportional to the intercept.

Given three epipolar lines with different d_i, we can define their ratio as $(d_1 - d_2)/(d_3 - d_2)$. This is an affine invariant and is the same in both images if the lines match. Given four such lines, we have two invariant ratios, which will yield two equations for s, s'.

2.1. Two Conics

Two conics in an image have four tangents with the same slope s, so they can be used in our method.

The tangents to a conic satisfy the equation of the line conic, the dual of the point conic. Denoting the regular (point) conic by the matrix A^{-1}, the line conic is $A = a_{ij}$.

To find tangents with slope s, we substitute the line vector in the line conic:

$$\mathbf{l}A\mathbf{l}^t = (s, 1, d) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} s \\ 1 \\ d \end{pmatrix} = 0$$

From the symmetry of A we obtain

$$a_{11}s^2 + 2a_{12}s + a_{22} + 2a_{13}sd + 2a_{23}d + d^2 = 0$$

The two solutions for this quadratic equation in d are

$$d^a_{1,2} = -a_{23} - a_{13}s \pm \sqrt{(a_{13}^2 - a_{11})s^2 + 2(a_{13}a_{23} - a_{12})s + a_{23}^2 - a_{22}}$$

To get rid of the square root it is more convenient to deal with the "width" of the conic, $d_1^a - d_2^a$, and with the d of the line going through its center, $d_c^a = \frac{1}{2}(d_1^a - d_2^a)$:

$$\frac{1}{2}(d_1^a - d_2^a)^2 = (a_{13}^2 - a_{11})s^2 + 2(a_{13}a_{23} - a_{12})s + a_{23}^2 - a_{22}$$

$$d_c^a = \frac{1}{2}(d_1^a + d_2^a)^2 = -a_{23} - a_{13}s$$

Similarly, for the other line conic B we have

$$\frac{1}{2}(d_1^b - d_2^b)^2 = (b_{13}^2 - b_{11})s^2 + 2(b_{13}b_{23} - b_{12})s + b_{23}^2 - b_{22}$$

$$d_c^b = \frac{1}{2}(d_1^b + d_2^b)^2 = -b_{23} - b_{13}s$$

We make the expressions for d_c^a, d_c^b independent of the unknown s by choosing a coordinate system in which $a_{13} = b_{13} = 0$. This is equivalent to choosing a y-axis that goes through the centers of both conics.

From the four values of d we can obtain two affine invariant ratios:

$$\frac{(d_1^a - d_2^a)^2}{(d_c^a - d_c^b)^2} = -\frac{a_{11}s^2 + 2a_{12}s - a_{23}^2 + a_{22}}{(a_{23} - b_{23})^2}$$

$$\frac{(d_1^b - d_2^b)^2}{(d_c^a - d_c^b)^2} = -\frac{b_{11}s^2 + 2b_{12}s - b_{23}^2 + b_{22}}{(a_{23} - b_{23})^2}$$

In the other image, we have again two (line) conics A', B'. The epipolar lines have a slope s' with respect to the local coordinate system. Therefore two invariants similar to those above exist, with a'_{ij}, b'_{ij}, s' replacing the primeless quantities above.

As discussed before, these invariant ratios are the same in both images, i.e.

$$\frac{a_{11}s^2 + 2a_{12}s - a_{23}^2 + a_{22}}{(a_{23} - b_{23})^2} = \frac{a'_{11}s'^2 + 2a'_{12}s' - {a'_{23}}^2 + a'_{22}}{(a'_{23} - b'_{23})^2} \tag{1}$$

$$\frac{b_{11}s^2 + 2b_{12}s - b_{23}^2 + b_{22}}{(a_{23} - b_{23})^2} = \frac{b'_{11}s'^2 + 2b'_{12}s' - {b'_{23}}^2 + b'_{22}}{(a'_{23} - b'_{23})^2} \tag{2}$$

We have thus obtained a system of two quadratic equations for the two unknowns s, s'. They can be solved by standard elimination methods. We obtain a quartic equation for s (or s'), leading to four solutions to the problem. As in the case of points, extra information such as more matching conics reduces the problem to a system of linear equations.

After finding the slopes, it is easy to find all the matches between epipolar lines. Two epipolar lines with parameters d, d' match each other if they are related by the invariant ratio

$$\frac{(d-d_c^a)}{(d-d_c^b)} = \frac{(d'-d_c^{a\prime})}{(d'-d_c^{b\prime})}$$

with all other quantities above now being known.

2.2. A Conic and Points

The method described above is easy to extend to a combination of a conic and points. Given points $\mathbf{x}_i$, they lie on the epipolar lines $(s, 1, d_i)$, i.e. they satisfy

$$\mathbf{l}_i\mathbf{x}_i = sx_i + y_i + d_i = 0$$

We thus have the following equations for d_i, d_i':

$$d_i = -sx_i - y_i, \qquad d_i' = -s'x_i' - y_i'$$

We can now define the two ratios

$$\frac{1}{2}\frac{(d_1^a - d_2^a)^2}{(d_1 - d_2)^2} = \frac{(a_{13}^2 - a_{11})s^2 + 2(a_{13}a_{23} - a_{12})s + a_{23}^2 - a_{22}}{(-s(x_1-x_2) - (y_1-y_2))^2}$$

$$\frac{d_c^a}{d_1 - d_2} = \frac{-a_{23} - a_{13}s}{-s(x_1-x_2) - (y_1-y_2)}$$

We can choose a coordinate system in which $x_1 - x_2 = 0$, eliminating s from the denominator. Equating the above two ratios to the ones in the other image we obtain

$$\frac{(a_{13}^2 - a_{11})s^2 + 2(a_{13}a_{23} - a_{12})s + a_{23}^2 - a_{22}}{(y_2-y_1)^2} = \frac{({a'_{13}}^2 - a'_{11})s'^2 + 2(a'_{13}a'_{23} - a'_{12})s' + {a'_{23}}^2 - a'_{22}}{(y'_2-y'_1)^2} \tag{3}$$

$$\frac{-a_{23} - a_{13}s}{y_2 - y_1} = \frac{-a'_{23} - a'_{13}s'}{y'_2 - y'_1} \tag{4}$$

We have two equations for s, s', which are easy to convert to one quadratic equation.

Given an additional point, we obtain an additional linear equation:

$$\frac{-s(x_1-x_3) - (y_1-y_3)}{y_2 - y_1} = \frac{-s'(x'_1-x'_3) - (y'_1-y'_3)}{y'_2 - y'_1} \tag{5}$$

Eqs. (4),(5) are a linear system in s, s'. They are equivalent to a four-point set, with the conic center (an affine invariant) being the fourth point. Given a system of many points and conics, we can solve an overdetermined linear system of equations either for s, s' or for s, s', s^2, s'^2, to obtain a more robust solution.

2.3. A Cubic

A cubic curve in 3D can be written in homogeneous coordinates as

$$\mathbf{X}(\tau) = \mathbf{C}_0 + \mathbf{C}_1\tau + \mathbf{C}_2\tau^2 + \mathbf{C}_3\tau^3$$

with $\mathbf{C}_i$ being constant vectors. It is projected into the images as two cubic curves $\mathbf{x}(t), \mathbf{x}'(t')$. The parameters in 3D and in the two images do not necessarily match. In fact, each parameter can change with three degrees of freedom, while the curve remains a cubic:

$$\tilde{t} = \frac{at+b}{ct+d}$$

If the parameters t_i at three points are known, the three constants a, b, c can be found and this determines the parameterization uniquely. Thus, given two images of a cubic with three matching points, the matching of any other point on the cubic is determined. (This in in fact true for any rational curve).

To find three matching points, we find three tangents to the cubic having slope s in the first image and s' in the other. Solving a cubic equation for the tangents, we obtain three values $d_i(s)$, $d_i'(s')$ in each image. These values obviously depend on the unknowns s, s'.

The contact points of these tangents match, and so they can be assigned the same values $\tilde{t}_i$ in both images, say 0,1,∞. This will determine the coefficients a, b, c for each image, and thus establish the same unique parameterization in both images. This solves the point correspondence problem between the two images.

To find the epipolar geometry, we can now select a fourth point, say $\tilde{t} = 2$, in each image and find the line with slope s (s') going through this point in the first (second) image. We have thus obtained four matching parallel lines in each image, depending on s, s'. As in the conic case, we have two invariant ratios, providing two equations for the unknowns s, s'.

Higher-order curves can be dealt with in a similar way, using three of their n matching tangents to establish the parameter correspondence. However, these tangents would not be very reliable in practice.

3. Reconstruction

Having found the epipolar geometry, we can reconstruct the 3D object.

3.1. Invariant Coordinates

The reconstruction is easier in term of a coordinate system which is common to the 2D and 3D spaces and is invariant in both.

We build this coordinate system using a set of reference points which are determined invariantly by the 3D object and its projections. We need four reference points in 3D, and three of their projections in each image. In the case of two 3D conics, we can use in 3D the four contact points $\mathbf{X}_i$ of the conics with the epipolar planes tangent to them. These $\mathbf{X}_i$ are projected onto the images as the contact points of the conics with the epipolar lines tangent to them. These tangents have known slopes s, s' and they touch the conics at known points $\mathbf{x}_i, \mathbf{x}'_i$. These reference points, both in 3D and 2D, are thus determined invariantly.

We can assign to the reference points in 3D the standard coordinates

$$(1,0,0,1), \quad (0,1,0,1), \quad (0,0,1,1), \quad (1,1,1,1) \tag{6}$$

In 2D we need only three points in each image. We can use $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$ in the first image and a slightly different subset, say $\mathbf{x}'_1, \mathbf{x}'_3, \mathbf{x}'_4$, in the other image. Both subsets can be assigned the coordinates

$$(1,0,1), \quad (0,1,1), \quad (1,1,1) \tag{7}$$

in their respective images. These choices make the projection matrices from 3D to 2D very simple in this "canonical", invariant system:

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \bar{P}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using this, the remaining 3D point is projected into the origin (0,0,1) in each image.

Any point in 3D can be expressed in terms of our 3D reference points as

$$\mathbf{U} = (U, V, W)$$

while in 2D we have

$$\mathbf{u} = (u, v), \qquad \mathbf{u}' = (u', v')$$

Using the projection matrices we find a very simple relation between the 3D and the 2D invariant coordinates:

$$u = u' = U, \qquad v = V, \qquad v' = W \tag{8}$$

That is, the first coordinate is common to both images and to the 3D space. It is related to the invariant ratio used earlier to find the epipolar geometry. The other coordinates are also common to 2D and 3D and can also be easily interpreted geometrically as ratios of planes and lines common to 3D and 2D.

The reconstruction can now proceed in two steps.

1) Given the curves in 2D Cartesian coordinates $\mathbf{x}, \mathbf{x}'$ in the images, transform them to the invariant coordinates $\mathbf{u}, \mathbf{u}'$. The transformation matrices between the two coordinate systems can be easily found from the linear equations

$$\mathbf{x}_i = j\mathbf{u}_i, \qquad \mathbf{x}'_i = j'\mathbf{u}'_i$$

where $\mathbf{u}_i, \mathbf{u}'_i$ are the invariant reference points as expressed in the invariant system, eq. (7), and j, j' are the transformation matrices. The coordinates $\mathbf{x}, \mathbf{x}'$ in the curve equations are then replaced by $j\mathbf{u}, j'\mathbf{u}'$ to obtain the curve equations in terms of $\mathbf{u}, \mathbf{u}'$.

2) Using the relations (8), the 2D invariant coordinates in the curve equations are replaced by the 3D ones. This creates 3D (general) cylinders arising from each image. These cylinders intersect to form our 3D curve.

There is also a 3D transformation matrix J analogous to the 2D ones, satisfying

$$\mathbf{X}_i = J\mathbf{U}_i$$

with $\mathbf{U}_i$ being the 3D reference points as expressed in the invariant system, eq. (6). However, since the $\mathbf{X}_i$ are unknown, we cannot find J. This matrix contains the unknown camera parameters. Therefore, we can reconstruct the 3D object only up to an arbitrary affine transformation J.

The projection matrices P, P' can be found from the "canonical" ones $\bar{P}, \bar{P}'$ defined earlier by

$$P = j\bar{P}J^{-1}, \qquad P' = j'\bar{P}'J^{-1}$$

Thus, if some of the coefficients in P, P' are known, we can recover some of the coefficients in J and therefore have more information about the 3D object.

3.2. Reconstructing Conics

We will now find the 3D conic equations in invariant 3D coordinates.

Carrying out the first step above, we move the conics in the images to the 2D invariant system. We have the (point) conics A, B in one image and A', B' in the other, satisfying

$$\mathbf{x}^t A \mathbf{x} = 0, \qquad \mathbf{x}'^t A' \mathbf{x}' = 0$$

and similarly for B, B'. Replacing $\mathbf{x}$ by $j\mathbf{u}$ and $\mathbf{x}'$ by $j\mathbf{u}'$ we obtain the conic matrices in the invariant system

$$\bar{A} = j^t A j, \qquad \bar{A}' = j'^t A' j'$$

which satisfy the equations

$$\mathbf{u}^t \bar{A} \mathbf{u} = \bar{a}_{11}u^2 + \bar{a}_{22}v^2 + 2\bar{a}_{12}uv + 2\bar{a}_{13}u + 2\bar{a}_{23}v + \bar{a}_{33} = 0$$

$$\mathbf{u}'^t \bar{A}' \mathbf{u}' = \bar{a}'_{11}u^2 + \bar{a}'_{22}v'^2 + 2\bar{a}'_{12}uv' + 2\bar{a}'_{13}u + 2\bar{a}'_{23}v' + \bar{a}'_{33} = 0$$

and similarly for $\bar{B}, \bar{B}'$. The coefficients $\bar{a}_i, \bar{a}'_i$ are not independent. There is only one independent quantity for each 2D conic. This is because in the invariant coordinate system, the contact points of the conic with the epipolar tangents have fixed coordinates, namely eq. (7) and (0,0,1). These also determine the slopes s, s' as fixed values. The conic has to pass through two of these contact points and be tangent there to the epipolar lines. This determines four out of the five conic parameters. Incidentally, one of the contact points in the invariant system is the origin, therefore one of the coefficients $\bar{a}_{33}, \bar{a}'_{33}$ equals 0, while the other equals 1.

Carrying out the second step, we replace the 2D coordinates u, v, v' in the equations above by the 3D coordinates U, V, W using eq. (8). We first deal with the conics A, A'. We obtain

$$\bar{a}_{11}U^2 + \bar{a}_{22}V^2 + 2\bar{a}_{12}UV + 2\bar{a}_{13}U + 2\bar{a}_{23}V + \bar{a}_{33} = 0$$

$$\bar{a}'_{11}U^2 + \bar{a}'_{22}W^2 + 2\bar{a}'_{12}UW + 2\bar{a}'_{13}U + 2\bar{a}'_{23}W + \bar{a}_{33} = 0$$

These are two (conic) cylinders, the axis of each being parallel to the direction of the projection onto the corresponding image. The intersection of these two cylinders forms our 3D conic. In general, two cylinders do not intersect in a plane conic. However, we have assumed *a priori* that our two images A, A' are formed by the same 3D plane conic, so this conic must lie on both cylinders.

We will now find the intersection of the cylinders. In general, it can be found by elimination methods. However, our case is particularly simple because the intersection is planar and the planes contain the 3D reference points (6). A plane can be generally written in the invariant coordinate system as

$$U = \alpha_1 V + \alpha_2 W + \alpha_3$$

Without loss of generality, we can assume that the plane Π^a, containing the conic arising from A, A', also contains the reference points $(1,0,0,1), (0,1,0,1)$. Substituting these points in the plane equation above we obtain $-\alpha_1 = \alpha_3 = 1$, so Π^a can be expressed as

$$\Pi^a: \qquad U = -V + \alpha W + 1 \tag{9}$$

α is an invariant characterizing the plane Π^a in the invariant coordinate system.

To find the invariant α, we eliminate U from the equations of the two cylinders above, using the plane equation (9). We obtain two plane conic equations in the variables V, W:

$$(\bar{a}_{11} - 2\bar{a}_{12} + \bar{a}_{22})V^2 + 2\alpha(-\bar{a}_{11} + \bar{a}_{12})VW + \alpha^2\bar{a}_{11}W^2$$

$$+2(-\bar{a}_{11} + \bar{a}_{12} - \bar{a}_{13} + \bar{a}_{23})V + 2\alpha(\bar{a}_{11} + \bar{a}_{13})W + \bar{a}_{11} + 2\bar{a}_{13} + \bar{a}_{33} = 0$$

$$\bar{a}'_{11}V^2 + 2(-\bar{a}'_{11}\alpha - \bar{a}'_{12})VW + (\bar{a}'_{11}\alpha^2 + 2\bar{a}'_{12}\alpha + \bar{a}'_{22})W^2$$

$$+2(-\bar{a}'_{11} - \bar{a}'_{13})V + 2((\bar{a}'_{11} + \bar{a}'_{13})\alpha + \bar{a}'_{12} + \bar{a}'_{23})W + \bar{a}'_{11} + 2\bar{a}'_{13} + \bar{a}'_{33} = 0$$

Since these equations represent the same conic in the plane Π^a, they must have the same coefficients. We thus obtain five equations (from the five conic coefficients) for the one unknown α. However, we have already seen that each cylinder has only one independent coefficient, so only one of these equations is meaningful. The rest are either identities or they are dependent on the first. We can see more directly that the 3D conic in Π^a has only one independent coefficient in our invariant coordinate system. The conic has to pass through two of the 3D reference points having the fixed coordinates (6), and be tangent there to the epipolar planes, which are known with fixed values from the corresponding epipolar lines. This fixes four of the conic coefficients.

We obtain α by equating the coefficients of VW:

$$\alpha = \frac{\bar{a}'_{12}}{\bar{a}_{11} - \bar{a}'_{11} - \bar{a}_{12}}$$

Substituting α in either of the conic equations above will give us the description of the 3D conic. All the coefficients are invariant, but only one is independent, as discussed above.

The other 3D conic can be found in a similar way. Its plane Π^b can be written as

$$\Pi^b : \qquad U = V + \beta W - \beta \tag{10}$$

and contains the reference points $(1,1,1,1), (0,0,1,1)$. Substituting this in the cylinders arising from B, B' we obtain the conics

$$\begin{gathered}(\bar{b}_{11} + 2\bar{b}_{12} + \bar{b}_{22})V^2 + 2\beta(\bar{b}_{11} + \bar{b}_{12})VW + \beta^2\bar{b}_{11}W^2 \\ +2(-\bar{b}_{11}\beta - \bar{b}_{12}\beta + \bar{b}_{13} + \bar{b}_{23})V + 2(-\bar{b}_{11}\beta^2 + \bar{b}_{13}\beta)W + \bar{b}_{11}\beta^2 - 2\bar{b}_{13}\beta + \bar{b}_{33} = 0 \\ \bar{b}'_{11}V^2 + 2(\bar{b}'_{11}\beta + \bar{b}'_{12})VW + (\bar{b}'_{11}\beta^2 + 2\bar{b}'_{12}\beta + \bar{b}'_{22})W^2 \\ +2(-\bar{b}'_{11}\beta + \bar{b}'_{13})V + 2(-\bar{b}'_{11}\beta^2 + \bar{b}'_{13}\beta - \bar{b}'_{12}\beta + \bar{b}'_{23})W + \bar{b}'_{11}\beta^2 - 2\bar{b}'_{13}\beta + \bar{b}'_{33} = 0\end{gathered}$$

We obtain β by equating the coefficients for VW:

$$\beta = \frac{\bar{b}'_{12}}{\bar{b}_{11} + \bar{b}_{12} - \bar{b}'_{11}}$$

Substituting this in the conic equations above, we obtain the conic in the plane Π^b. Again, all the coefficients are invariants, with only one being independent.

In sum, each conic has two invariants associated with it: the coordinate of the plane on which it lies (α or β), and one invariant conic coefficient (or any one function of the coefficients). Altogether, we thus have four 3D invariants. This is what we concluded earlier from a simple counting argument.

We can find a geometric construction of the conic invariants by a method we used earlier. First we find the line of intersection of the two planes Π^a, Π^b. In each plane we can now draw tangents to the conics that are parallel to this intersection line. The ratio of distances of the tangents from the intersection line is an affine invariant of the conic.

The above method is easily applied to a combination of a conic and points. Two of the invariant reference points in 3D will be determined by the conic as before, and the other two will be the given points. The conic is reconstructed using the same equations used above (with either A or B). The system will have two 3D invariants: the plane coordinate α (or β) and the invariant of the conic lying in it.

4. Conclusion

We have reconstructed 3D objects, such as a pair of non-coplanar conics, from uncalibrated images, up to an affine transformation. No knowledge of the epipolar geometry was needed; it was recovered from the images. The method is based on using invariants of epipolar lines that are equal in the two images when these lines match. This invariance is used to find the epipolar geometry. We have built a coordinate system which is common to 3D and 2D and is invariant in both. This invariant system has made the reconstruction very simple and general. The affine invariants of the 3D objects are obtained very easily in this system. Since many curves can be approximated by or segmented into conics, the method can be useful in many applications.

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There has been considerable work recently on the problem of reconstruction of 3D points sets from two images, taken by uncalibrated cameras. However, the point correspondence has to be given. Here we deal with reconstruction of curves rather than points. While we need the correspondence between curves, this is an easier problem because curves are far fewer and more distinctive than points. We derive a simple and general reconstruction method, based on an invariant coordinate system. We then apply it to non-coplanar conics and to combinations of a 3D conic with points. 3D cubics are also discussed. Unlike previous work, we do not need to know the epipolar geometry; we recover it from the images.

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